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ON THE STABILITY OF EQUILIBRIUM STATES OF THE NON-HOLONOMIC SYSTEMS*

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SUMMARY

It is shown in the present paper that a non-holonomic system has the peculiarity that its equilibrium states cannot be isolated; they form a variety, whose dimensionality is equal to the number of equations of non-holonomic liaisons. This peculiarity is the one that conditions the presence of zero radicals in the characteristic equation**. A theorem is formulated about the asymptotic stability of manifold states of equilibrium. The theory, here expounded, is illustrated by an example.

author

• • •

1. - Assume that upon a system with a Lagrange function $L = L(q_1, \ldots, q_n, \dot{q}_1 \ldots, \dot{q}_n)$ and with generalized forces $Q_\beta = Q_\beta(q_1, \dot{q})$ ($\beta = 1, 2, \ldots n$) non-holonomic liaisons are superimposed, which are determined by m ($m < n$) equations ***

$$\omega_{\alpha\beta}(q_1, \ldots, q_n)\,\dot{q}_\beta = 0 \quad (\alpha = 1, 2, \ldots, m;\ \beta = 1, 2, \ldots, n). \tag{1}$$

The equations of motion with undetermined multipliers

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\beta} - \frac{\partial L}{\partial q_\beta} = Q_\beta + \lambda_\alpha \omega_{\alpha\beta} \tag{2}$$

form together with (1) a system of $(n + m)$ equations relative to $q_1, q_2, \ldots, q_n, \lambda_1, \lambda_2, \ldots, \lambda_m$. It follows from (1) and (2) that the states of equilibrium of the non-holonomic system are determined by n equations

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* OB USTOYCHIVOSTI SOSTOYANIY RAVNOVESIYA NEGOLOMNYKH SISTEM.
** (see addendum 1 at the end of the paper).
*** Here and further summing up is effected by twice recurring indices, and the dot above the letter denotes a derivative in time.

$$\frac{\partial L}{\partial q_\beta} + Q_\beta + \lambda_\alpha \omega_{\alpha\beta} = 0 \qquad (3)$$

relative to (n + m) unknowns $q_1, \ldots, q_n, \lambda_1, \ldots, \lambda_m$. By the strength of the above we shall have in the general case a variety of equilibrium states forming in the n-th dimensional space of configurations, a surface O_m of dimensionality m. In fact, expressing with the aid of equations (3) the generalized coordinates $q_1, \ldots, q_n$ through $\lambda_1, \ldots, \lambda_m$, we shall obtain the surface in the parametric representation $q_\beta^0 = q_\beta^0(\lambda_1, \ldots, \lambda_m)$. Note, that in coherent problems we may encounter a case when the equations (3) are not independent, which has as a corollary the increase in the dimensionality of the variety of equilibrium states .

2.- Excluding from equations (2) the undeterminate multipliers $\lambda_1, \ldots, \lambda_m$, we shall write the equations of motion (1), (2), of the non-holonomic system in the normal form

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_{2n-m}) \; (i = 1, 2, \ldots, 2n - m), \qquad (4)$$

where we denoted by x_i the variables $q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_{n-m}$. Assume that in the phase space $(x_1, x_2, \ldots, x_{2n-m})$ the surface O_m is determined by the equations $x_i^0 = x_i^0(u_1, \ldots, u_m)$ $(i=1, 2, \ldots, 2n-m)$. Alongside with the variables $u_1, \ldots, u_m$ we shall introduce new variables $v_1, \ldots, v_{2(n-m)}$ by means of the correlations

$$x_i = x_i^0(u_1, \ldots, u_m) + \gamma_{ij}(u_1, \ldots, u_m)\, v_j.$$

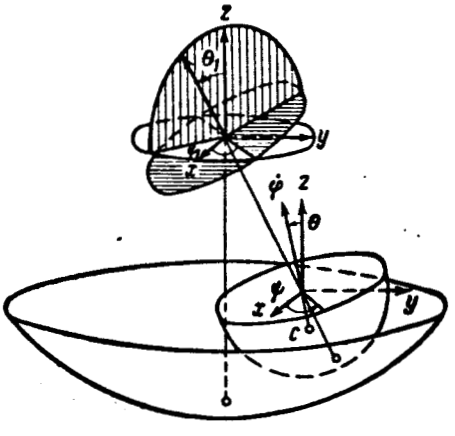

Fig. 1

In the new variables, the equations (4) will be written in the form

$$\dot{u}_i = g_i(u, v), \quad \dot{v}_j = g_j(u, v). \qquad (5)$$

Linearizing the equations of motion (5) in the vicinity of the surface O_m of equilibrium states and expanding the right-hand parts of (5) in series by small values of v, we shall obtain

$$\begin{aligned} \dot{u}_i &= a_i(u_1, \ldots, u_m) + a_{ij}(u_1, \ldots, u_m)\, v_j + O(\|v\|^2) + \ldots, \\ \dot{v}_j &= b_j(u_1, \ldots, u_m) + b_{jk}(u_1, \ldots, u_m)\, v_k + O(\|v\|^2) + \ldots, \end{aligned} \qquad (6)$$

where a_i and b_i are zero, since $v = \dot{v} = \dot{u} = 0$ on the surface O_m. In correspondence with this the characteristic equation of the system (6) for

any point of the surface O_m, has the form:

$$p^m \,|\, b_{ij} - p\delta_{ij} \,| = 0. \tag{7}$$

3.- Let us now pass to the clarification of the conditions of stability. It follows from the preceding, that the study of the stability of equilibrium states of a non-holonomic system makes sense only relative to small deflections from the surface O_m. At the same time, it is natural to consider the second group of equations (6) independently from the first group, provisionally interpreting the variables $u_1, \ldots, u_m$ as parameters. The characteristic polynomial of this auxiliary system differs from the determinant (7) only by the absence of the multiplier p^m.

Assume that in certain region G of values of $u_1, \ldots, u_m$, the equilibrium state, $v = 0$, of the system of equations

$$\dot{v}_j = b_{jk}\,(u_1, \ldots, u_m)\, v_k \tag{8}$$

is asymptotically stable, so that * $\|v\| < M \|v^0\| e^{-\sigma t}$, where $\sigma > 0$, v^0 are the initial values of variables v; then, the following theorem takes place.

THEOREM.- Let the initial values $u_1^0, \ldots, u_m^0, v_1^0, \ldots, v_{2(n-m)}^0$ be such that the values $u_1^0, \ldots, u_m^0$ lie inside the region G of the asymptotic stability, and $v_1^0, \ldots, v_{2(n-m)}^0$ be sufficiently small. Then, by the strength of equations (5)

$$\lim_{t \to +\infty} v_j(t) = 0, \quad \lim_{t \to +\infty} u_i(t) = u_i^*, \tag{9}$$

where $u_i^* \in O_m$, but, generally $u_i^* \neq u_i^0$. At the same time, for the variables $v_j(t)$ the following estimate takes place

$$\|v(t)\| < M' \|v^0\| e^{-\sigma' t}, \tag{10}$$

where $\sigma > \sigma' > 0$.

DEMONSTRATION. Let us write the equations (6) in the form

$$\begin{aligned} u_i &= \{a_{ij}\,(u_1, \ldots, u_m) + \Delta a_{ij}\}\, v_j, \\ \dot{v}_j &= \{b_{jk}\,(u_1, \ldots, u_m) + \Delta b_{jk}\}\, v_k, \end{aligned} \tag{11}$$

where $|\Delta a_{ij}| < \varepsilon$ & $|\Delta b_{jk}| < \varepsilon$, provided the condition

$$\|u - u^0\| < \delta(\varepsilon), \quad \|v\| < \delta(\varepsilon) \quad [\delta(\varepsilon) > 0] \tag{12}$$

is satisfied.

So long as the inequalities (12) are not satisfied, and at sufficiently small $\delta = \delta^*$ for the solution of the equations (11), the following

* [see next page]

estimate takes place [6]:

$$\|v(t)\| < M' \|v^0\| e^{-\sigma' t}, \tag{13}$$

and that is why

$$\|\dot{u}\| < N \|v^0\| e^{-\sigma' t},$$

$$\|u(t)\| < \frac{N}{\sigma} \|v^0\|. \tag{14}$$

Assume that for the selected value $\delta = \delta^*$ the inequality

$$\|v^0\| < \min\left(\frac{\delta^*}{2M}, \frac{\delta^* \sigma}{2N}, \frac{\delta^*}{2}\right). \tag{15}$$

is fulfilled.

At the initial moment of time $t = 0$ the conditions (12) are fulfilled for $\delta = \delta^*/2$, and therefore, by the strength of continuous dependence of the solution on time t, they will be fulfilled in the course of a certain moment of time Δt_0 and, consequently, the estimates (13) and (14) will take place during the same time interval. By the strength of these estimates and of the inequality (15), by the time lapse Δt_0 the quantity $v(\Delta t_0)$ will satisfy the inequalities (12) with $\delta = \delta^*/2$, and that is why these inequalities are fulfilled during a certain time interval $\Delta t_0 + \Delta t_1$. Extending these reasonings, we shall establish that the estimates (13) and (14) take place for all $\underline{t}$'s since $\Delta t_s > \tau > 0$, inasmuch as at the moment of time $\Delta t_0 + \ldots \ldots + \Delta t_{s-1}$ the inequalities (13) and (14) actually take place. From the fulfillment of the estimates (13) and (14) for all $\underline{t}$'s follows the corroboration of the theorem.

As an example let us consider the motion of an axisymmetrical body, bounded underneath by a spherical surface of radius R, which may be rolling without slippage through a spherical cup of radius R_1. The body's mass center is situated at the distance l from the center of its spherical surface. In accordance with Fig. 1 the generalized coordinates of the body are the angles $\theta, \psi, \varphi, \theta_1, \psi_1$. The Lagrange function

$$L = \tfrac{1}{2} m (R_1 - R)^2 (\dot{\theta}_1^2 + \dot{\psi}_1^2 \sin^2 \theta_1) + \tfrac{1}{2} (A + ml^2)(\dot{\theta}^2 + \dot{\psi}^2 \sin^2 \theta) +$$
$$+ ml(R_1 - R)\{\dot{\psi}\dot{\psi}_1 \sin\theta \sin\theta_1 \cos(\psi - \psi_1) + \dot{\theta}\dot{\theta}_1 \cos\theta \cos\theta_1 \cos(\psi - \psi_1) +$$
$$+ \sin\theta \sin\theta_1] + \dot{\theta}\dot{\psi}_1 \cos\theta \sin\theta_1 \sin(\psi - \psi_1) - \dot{\psi}\dot{\theta}_1 \sin\theta \cos\theta_1 \sin(\psi - \psi_1)\} +$$
$$+ \tfrac{1}{2} C (\dot{\varphi} + \dot{\psi}\cos\theta)^2 + mg[(R_1 - R)\cos\theta_1 + l\cos\theta].$$

where m is the mass, A, C are the central moments of inertia, g is the gravitation acceleration.

[From page 3]. * The symbol v denotes a vector with components n_j, and the symbol $\|v\|$ — $\|v\| = [v_1^2 + \ldots \ldots + v_{2^{(n-m)}}^2]^{1/2}$.

The dissipation function is

$$\Phi = {}^1/_2 h\left(\frac{R_1}{R} - 1\right)^2 (\dot\theta_1^2 + \dot\psi_1^2 \sin^2\theta_1) + {}^1/_2 h_1 \{\dot\psi \cos\theta_1 - \dot\theta \sin\theta_1 \sin(\psi - \psi_1) + \\ + \dot\varphi [\sin\theta \sin\theta_1 \cos(\psi - \psi_1) + \cos\theta \cos\theta_1]\}^2,$$

where $h > 0$, $h_1 > 0$ are the coefficients of viscous friction of rolling and rotating. The condition of rolling without slippage leads to two nonholonomic liaison equations

$$(R_1 - R)\dot\psi_1 \sin\theta_1 + R\dot\theta \cos\theta_1 \sin(\psi - \psi_1) + R\dot\psi \sin\theta_1 + \\ + R\dot\varphi [\cos\theta \sin\theta_1 - \sin\theta \cos\theta_1 \cos(\psi - \psi_1)] = 0;$$

$$(R_1 - R)\dot\theta_1 + R\dot\theta \cos(\psi - \psi_1) + R\dot\varphi \sin\theta \sin[\psi - \psi_1) = 0.$$

Composing the equations of motion (2), where the generalized forces will be $Q_\beta = -\partial\Phi/\partial\dot q_\beta$, we shall then obtain the following equilibrium equations:

$$\lambda_1 \sin\theta_1 = 0; \quad mg \sin\theta_1 = \lambda_2; \quad \lambda_1 \sin\theta_1 = 0; \\ mgl \sin\theta = R\lambda_1 \cos\theta_1 \sin(\psi - \psi_1) + R\lambda_2 \cos(\psi - \psi_1); \qquad (16)$$

$$\lambda_1 [\cos\theta \sin\theta_1 - \sin\theta \cos\theta_1 \cos(\psi - \psi_1)] + \lambda_2 \sin\theta \sin(\psi - \psi_1) = 0,$$

where λ_1, λ_2 are undetermined multipliers. It follows from the equations (16) that the surface O_m of equilibrium states is determined by the equations

$$\psi = \psi_1,$$

$$l \sin\theta = R \sin\theta_1$$

and it is three-dimensional.

The characteristic equation is brought to the form

$$p^3 (a_0 p^2 + a_1 p + a_2)(b_0 p^3 + b_1 p^2 + b_2 p + b_3) = 0,$$

and its coefficients denote certain functions of system's parameters. For the values $R_1 > R > l > 0$ the stability region of system's equilibrium states' variety is determined by the inequalities $b_3 > 0$, $b_1 b_2 - b_0 b_3 > 0$.

The Fig. 2 indicates the boundaries of the stability region at $R_1 = 4R$ for a uniform hemisphere, where

$$\delta = h\,(mR\sqrt{gR})^{-1},$$

$$\delta_1 = h_1\,(mR\sqrt{gR})^{-1}.$$

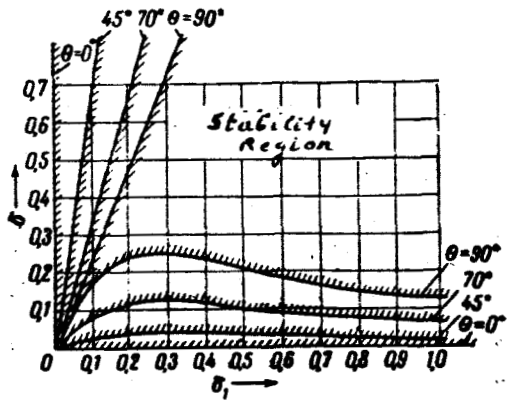

Fig. 2

**** THE END ****

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ADDENDUM

** [From page 1]. The question of stability of equilibrium states of non-holonomic states was considered in Uitteker's and other works [1 — 5]. The methods of stability investigation proposed in these works and also the viewpoints on the nature of zero radicals do not agree with one another. As is well known, Uitteker considered the integration of linearized equations of non-holonomic liaisons as possible, after which the distinction between the non-holonomic and holonomic systems disappeared. Having eliminated the inaccuracy in Uitteker's reasonings, Bottema has shown that contrary to holonomic system, the characteristic determinant of a non-holonomic system is asymmetric and the characteristic equation has zero radicals. Bottema reached the conclusion that there takes place a critical case of the stability theory of an isolated equilibrium state. M.A. Aiserman and F.R. Gantmacher have shown that this case is reduced to the particular one studied by Lyapunov and Malkin.

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